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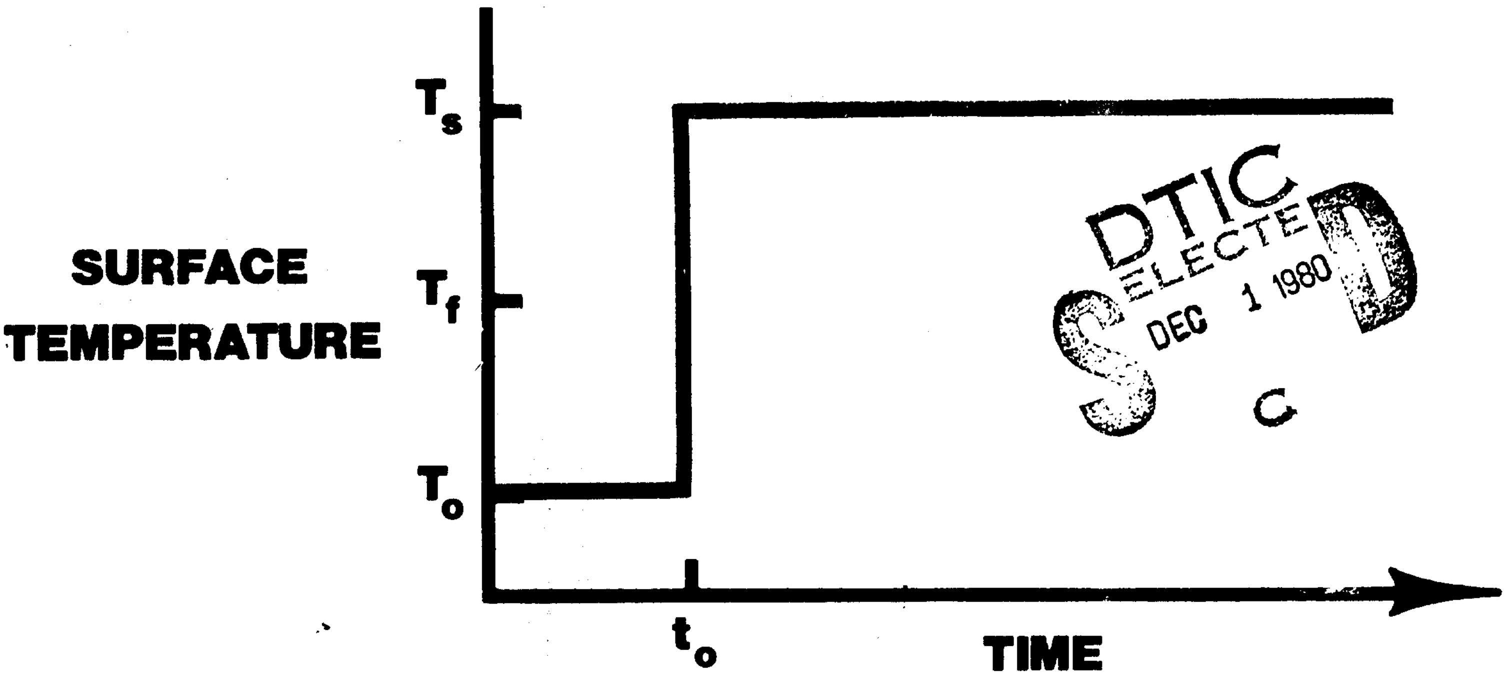
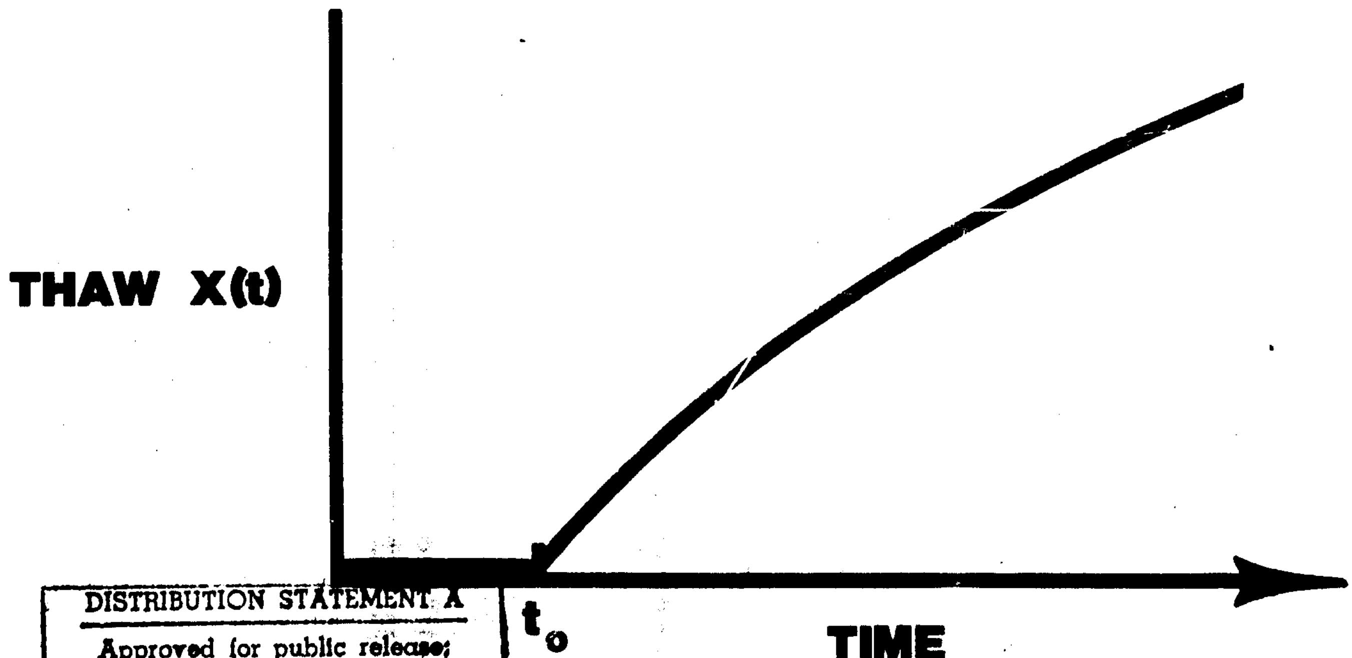
REPORT 80-22

The Neumann solution applied to soil systems





Cover: Freezing or thawing of a homogeneous soil (the Neumann solution).

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The Neumann solution applied to soil systems

V.J. Lunardini

October 1980

UNITED STATES ARMY
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The only complete, analytic solution for conduction problems with phase change is the Neumann solution. The Neumann solution is valid for phase change in a semi-infinite, homogeneous medium with a step change in surface temperature, starting from an initial temperature which can be different than or equal to the fusion temperature of the medium. The Neumann solution, when applied to soils, forms the basis of a number of formulae for calculating the depths of freezing or thawing. Widely used graphs were previously developed that are valid only when the ratios of the thermal conductivities and thermal diffusivities of the frozen and thawed soils are unity. In this report general charts, applicable to any property ratios, are developed. The figures have been drawn specifically for soil systems, but they are applicable to any material with appropriate property ratios.

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PREFACE

This report was prepared by Dr. Virgil Lunardini, Mechanical Engineer, Applied Research Branch, Experimental Engineering Division, U.S. Army Cold Regions Research and Engineering Laboratory.

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NOMENCLATURE

b	$\sqrt{\kappa_{ft}} = \sqrt{\kappa_f/\kappa_t}$
C	volumetric specific heat or heat capacity
I_f, I_t	air index for freeze and thaw
I_s	surface index
k	thermal conductivity
L	volumetric latent heat
n	ratio of surface to air index
p, q, r	property ratios defined in text
R	ratio of thermal conductivity of ice to water
t	time
T_f, T_o, T_s	fusion, initial, and surface temperatures
x	volumetric fractions of soil solids, liquids, gases
X	depth of phase change
α	$(T_f - T_o)/(T_s - T_f)$
γ	phase change parameter
θ	length of thaw or freeze season
κ	thermal diffusivity
λ	phase change parameter
μ	$(C_t/2L)(T_s - T_f)$ thawing $(C_f/2L)(T_f - T_s)$ freezing

Subscripts

f	frozen
g	gas
i	ice
ℓ	liquid
s	solid
t	thawed
w	water

THE NEUMANN SOLUTION APPLIED TO SOIL SYSTEMS

V.J. Lunardini

INTRODUCTION

One of the most common problems facing engineers in the cold climate regions of the world is the need to estimate the depth of freeze or thaw of a soil system. Despite the importance of moving boundary problems of conduction with phase change, there are only a handful of exact solutions. One of these solutions, which includes both latent and sensible heat of the soil mass, is that of Neumann (ca. 1860) which was generalized by Carslaw and Jaeger (1959).

At the start of a freezing process, a semi-infinite medium, initially in the thawed state at T_o, suddenly undergoes a step change in surface temperature to T_s which is less than the fusion temperature T_f. The solution for the phase change depth X is given by

$$X = 2\gamma\sqrt{\kappa_f t}\ . \tag{1}$$

The phase change parameter γ can be obtained from the following equation:

$$\frac{e^{-\gamma^2}}{\mathrm{erf}\,\gamma} - \frac{k_t}{k_f}\frac{(T_o - T_f)}{(T_f - T_s)}\frac{b\,e^{-(b\gamma)^2}}{\mathrm{erfc}\,b\gamma} = \frac{L\gamma\sqrt{\pi}}{C_f(T_f - T_s)} \tag{2}$$

where

$$\mathrm{erf}\,\gamma = \frac{2}{\sqrt{\pi}}\int_0^\gamma e^{-x^2}\,dx$$

$$\mathrm{erfc}\,\gamma = 1 - \mathrm{erf}\,\gamma.$$

Carslaw and Jaeger (1959) present solutions to the transcendental equation for γ in the case of pure water and ice. Equation 1, the Neumann equation, is used as the basis for many phase change studies and is applicable to engineering problems. Berggren (1943) was apparently the first to actually apply the Neumann solution to soil phase change problems. Aldrich and Paynter (1953) later used the Stefan form of the phase change solution to arrive at the modified Berggren equation. Equation 1 can be changed to the Stefan form as

$$X = \lambda\sqrt{(2k_f/L)(T_f - T_s)t} \tag{3}$$

where λ, which replaces γ, can be determined from the exact solution. Stefan (1891) originally solved a similar problem for the growth of sea ice when the sensible heat to latent heat ratio was small and the water was at the freezing temperature. Equation 3 reduces to the Stefan equation when λ equals one; hence the name "Stefan form."

The surface temperature of a soil system does not normally remain constant during the freeze season and the surface index I_s is often used.

$$X = \lambda\sqrt{(2k_f/L)\,I_s}\ . \tag{4}$$

The surface index is defined as

$$I_s = \int_0^\theta [T_f - T_s(t')]\,dt' = (T_f - \overline{T}_s)\theta \tag{5}$$

where θ is the length of the freeze season.

Thus an average constant surface temperature $\overline{T}_s$ can be calculated for the season to be used in eq 1 or eq 3, if I_s is known. Unfortunately the surface index is rarely available for a location; however, the air temperature index I_f is usually tabulated and I_s can be replaced with the n-factor, n defined as

$$n = I_s/I_f \quad \text{or} \quad I_s/I_t\ . \tag{6}$$

The quantity n is the relation between the air index and the surface index. A procedure for obtaining a value at a given site is given by Lunardini (1978).

Finally, the modified Berggren equation is written as

$$X = \lambda\sqrt{(2k_f\,I_f\,n)/L}\ . \tag{7}$$

Berg and Aitken (1973), among many others, have shown that the modified Berggren equation (eq 7) gives good results for seasonal phase change depths even if the surface temperature varies with time. The coefficient

γ can be found by equating eq 1 and 3 and substituting into eq 2. This leads to the following equation for γ:

$$\frac{e^{-\lambda^2\mu}}{\operatorname{erf}(\lambda\sqrt{\mu})} - \frac{p\alpha e^{-q\lambda^2\mu}}{\operatorname{erfc}(r\lambda\sqrt{\mu})} = \frac{\lambda}{2}\sqrt{\pi/\mu}. \qquad (8)$$

The parameters α and μ take into account the soil temperatures, specific heat and latent heat. The parameter μ is one-half of the Stefan number which is the ratio of the sensible heat and latent heat for a soil system. For small values of μ and α, it can be expected that λ will be nearly one and eq 7 will reduce to the Stefan equation.

The quantities p, q and r are ratios of the thermal properties of the soil system for the frozen and thawed states

$$p = (k_t/k_f)\sqrt{\kappa_f/\kappa_t}$$

$$q = \kappa_f/\kappa_t$$

$$r = \sqrt{\kappa_f/\kappa_t}\,.$$

These relations are all for the freezing case. Aldrich and Paynter (1953) used the relations

$$\alpha = (T_o - T_f)/(T_f - T_s)$$

$$\mu = (C_f/L)(T_f - T_s)$$

and noted that calculations with typical soil properties indicated $\kappa_f/\kappa_t \simeq 1.0$, $C_t/C_f \simeq 1$, and thus $k_t/k_f \simeq 1.0$. They then solved eq 8 with $p = q = r = 1$ and obtained a widely used graph for λ (see Sanger 1969). Actually, this procedure is only valid when the water content of a soil is zero. Nixon and McRoberts (1973) made a parametric study of eq 2, but presented a graph of λ valid only for $r = \sqrt{\kappa_{ft}} = 1.43$, which was said to represent most soil conditions.

SOIL THERMAL PROPERTIES

It is clear that p, q and r will vary for different soil systems, but a relatively simple procedure can be used to generate these functions for any soil. Gold and Lachenbruch (1973) noted that the weighted, geometric mean for the thermal conductivity of a mixture gives results that are often as good as more complicated methods. The thermal conductivity of a soil can then be expressed as

$$k = (k_s)^{x_s}(k_\ell)^{x_\ell}(k_g)^{x_g} \qquad (9)$$

where k_s, k_ℓ and k_g are the thermal conductivities of the solid, liquid, and gaseous phases; x_s, x_ℓ and x_g are the volume fractions of the solid, liquid, and gaseous phases. For soil systems, the thermal conductivity of the solids and gases will not vary significantly as phase change occurs (Kersten 1949). There will be only a small error if it is assumed that the frozen state contains only ice with no unfrozen water. Thus the ratio of the frozen to unfrozen conductivities of the soil mixture can be related to the thermal conductivity of ice and water as follows:

$$k_t/k_f = (k_w/k_i)^{x_\ell} \qquad (10)$$

where k_w, k_i are the thermal conductivities of water and ice. The volumetric specific heat for the system may be expressed for the thawed and frozen states as follows:

$$C_t = C_{st}(1-x_\ell) + C_w x_\ell \qquad (11)$$

$$C_f = C_{sf}(1-x_\ell) + C_i x_\ell \qquad (12)$$

where C_{st}, C_{sf} = specific heats of unfrozen and frozen soil solids. The neglect of the gas phase is insignificant since the density of the gas (air) is low. It is fortunate that the specific heats of different soil solids and ice are all similar in magnitude. For example, the volumetric specific heat of organic solids is about 2300 kJ/m^3 K, for mineral solids it is 1760, while for ice it is 1920 (see Lunardini 1971). The properties of the frozen materials are evaluated at 25°F (269 K) while the thawed values are at 40°F (277.4 K). If one assumes that the values for the solids, except for ice, change little through the phase change then

$$C_t/C_f = 1 + |(C_w/C_i) - 1|\, x_\ell \qquad (13)$$

or

$$C_t/C_f = 1 + 1.023\, x_\ell\,.$$

These relations indicate that the soil property ratios may vary as

$$0.25 < k_t/k_f < 1.0$$

$$1.0 < C_t/C_f < 2.1$$

$$0.13 < \kappa_t/\kappa_f < 1.0.$$

The property values to use in eq 8 can then be expressed as simple functions of the soil water content

$$q = R^{x_\ell}\,(1 + 1.023\, x_\ell)$$

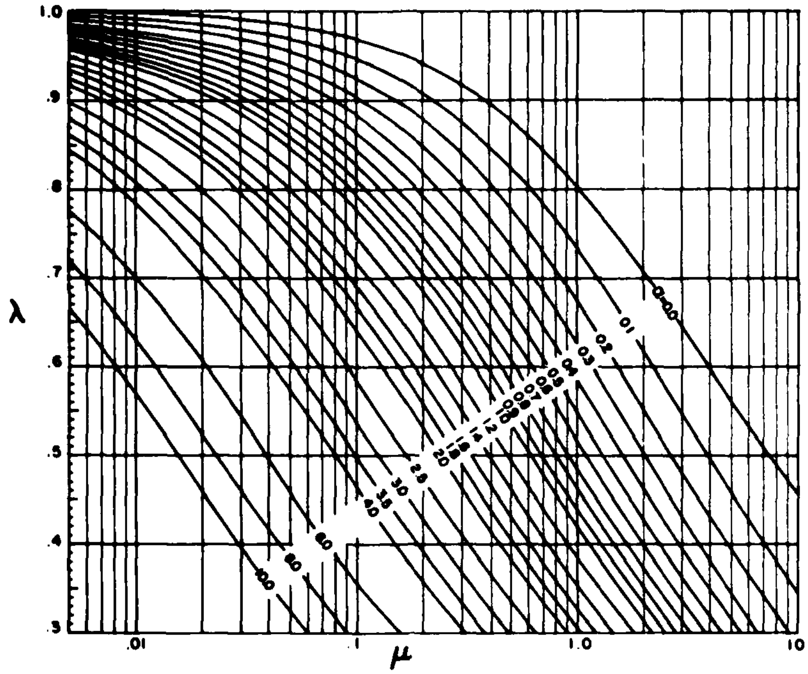

Figure 1. Freeze or thaw, Neumann equation parameter, $x_\mathrm{q} = 0.0$.

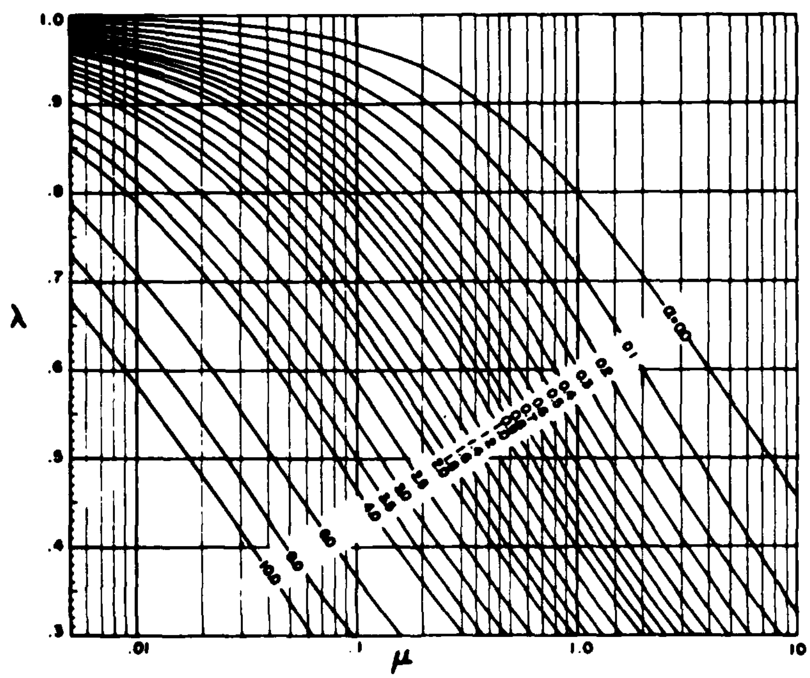

Figure 2. Freezing case, Neumann equation parameter, $x_\mathrm{q} = 0.4$.

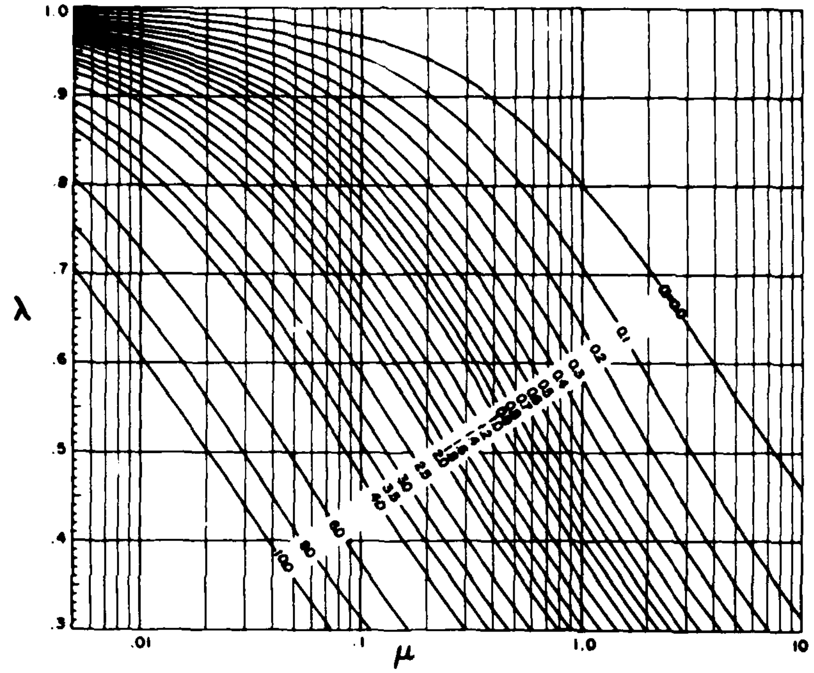

Figure 3. Freezing case, Neumann equation parameter, x_{ξ} = 0.8.

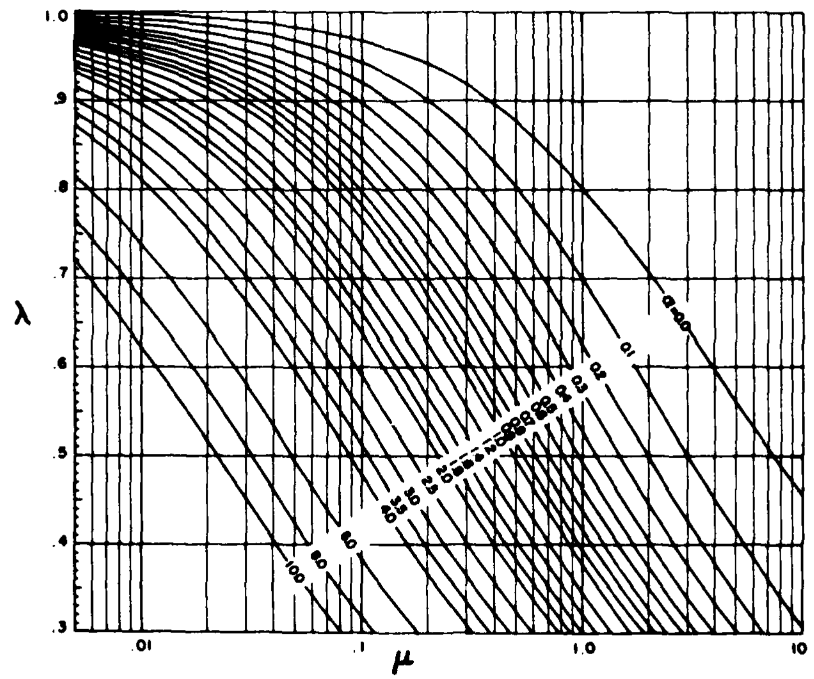

Figure 4. Freezing case, Neumann equation parameter, x_{ξ} = 1.0.

$$r = \sqrt{q}$$

$$p = r/R^{x_\ell}.$$

The ratio of the thermal conductivity of ice to water R is about 3.89. Equation 8 can be solved numerically to find the roots, which are the values of λ. Figures 1 through 4 give the values of λ to use in eq 3 or eq 7 for the freezing case.

THAW CASE

At first glance, it might appear that the same relations could be used for either the thawing or the freezing case. This, however, is not true. In the thawing case, the medium is initially frozen at T_0 and energy must be conducted through the thawed layer from the phase change interface. Since the thermal conductivity of the thawed region is considerably less than that of the frozen, the heat flow will be reduced, even with the same temperature gradient. However, the general form of the equation will be the same; after making appropriate changes for the property values. The thaw depth is expressed as

$$X = \lambda\sqrt{(2k_t\, I_t\, n)/L} \qquad (14)$$

where λ is again given by eq 8 but the parameters α and μ are now $\alpha = (T_f - T_0)/(T_s - T_f)$ and $\mu = (C_t/2L)(T_s - T_f)$. The functions p, q and r all change because of the property changes of the thawed and frozen states.

$$q = 1/[R^{x_\ell}(1 + 1.023\, x_\ell)].$$

$$r = \sqrt{q}.$$

$$p = r\, R^{x_\ell}.$$

The λ values for thawing are now given in Figures 5 through 7. Notice that when $x_\ell = 0$ the λ values are the same for freezing or thawing (Fig. 1), which is the Aldrich (1953) case and the Sanger (1969) graph. The charts for λ are for the exact solution of the Neumann problem with property values typical of soil systems.

DISCUSSION

A comparison of the value of λ obtained with Figures 1 through 7 and the value obtained from the chart given in Sanger (1969) shows that the values can differ by at least ±10%. Also, for the same values of x_ℓ, α and μ, the value of λ differs by ±10% when the freezing and thawing cases are considered. Since Sanger's

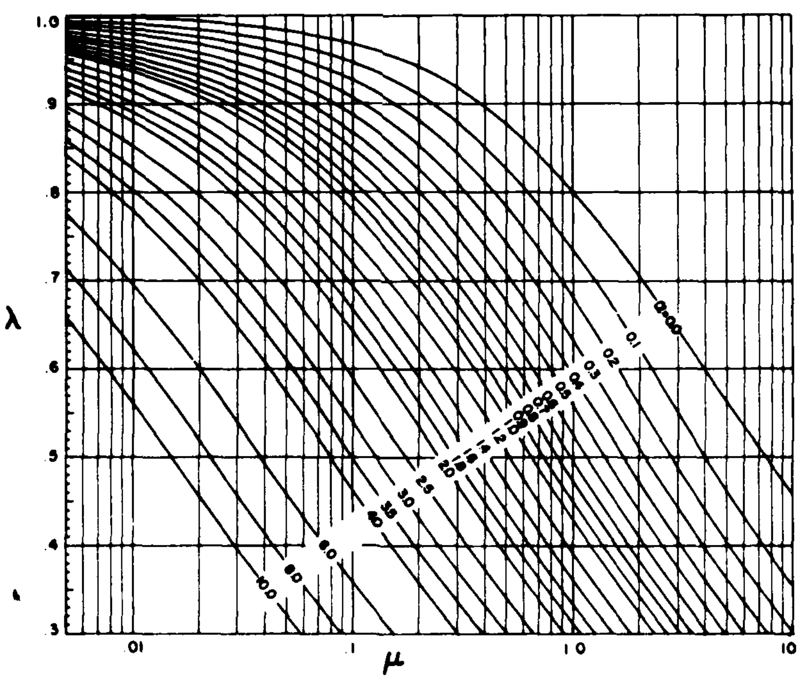

Figure 5. Thawing case, Neumann equation parameter x_ℓ = 0.4.

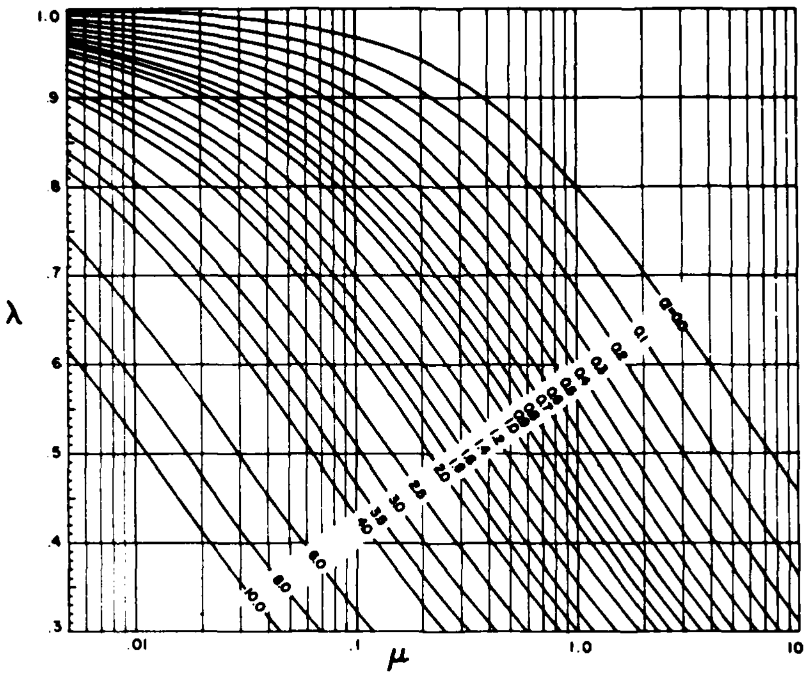

Figure 6. Thawing case, Neumann equation parameter, $x_{\ell} = 0.8$.

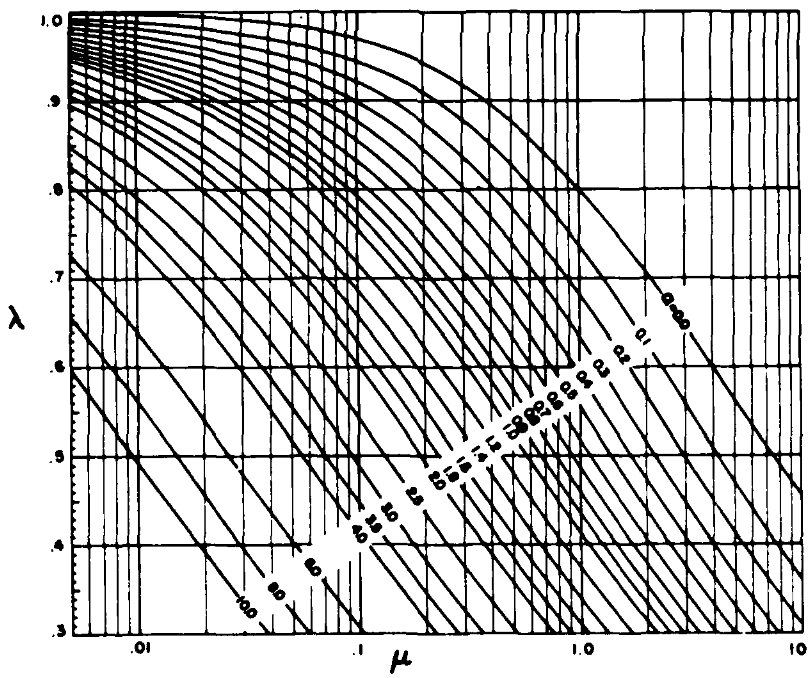

Figure 7. Thawing case, Neumann equation parameter, $x_{\ell} = 1.0$.

chart uses property ratios of one, there is no distinction between thawing and freezing. While these variations are not extreme they will lead to the same percentage differences for the calculated depths of freeze or thaw. No additional computational work is involved and it seems desirable to use the graphs presented here.

If the modified Berggren equation is programmed into a computer then eq 8 would be used directly to generate the λ values. Equation 8 can be solved by any convenient scheme such as iteration or Newton's method. The values for Figures 1 through 7 were obtained with Newton's method. The quantities p, q and r can be calculated directly if the thermal property ratios are specified or by using the functional dependence upon the volumetric water content derived in this report.

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